A Complex of Incompressible Surfaces for handlebodies and the Mapping Class Group.

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January 19, 2013

Abstract

For a genus g handlebody H_g a simplicial complex, with vertices being isotopy classes of certain incompressible surfaces in H_g, is constructed and several properties are established. In particular, this complex naturally contains, as a subcomplex, the complex of curves of the surface ∂H_g. As in the classical theory, the group of automorphisms of this complex is identified with the mapping class group of the handlebody. [1]

1 Definitions and statements of results

For a compact surface F, the complex of curves $\mathcal{C}(F)$, introduced by Harvey in [6], has vertices the isotopy classes of essential, non-boundary-parallel simple closed curves in F. A collection of vertices spans a simplex exactly when any two of them may be represented by disjoint curves, or equivalently when there is a collection of representatives for all of them, any two of which are disjoint. Analogously, for a $3-$manifold M, the disk complex $\mathcal{D}(M)$ is defined by using the proper isotopy classes of compressing disks for M as the vertices. It was introduced in [12], where it was used in the study of mapping class groups of $3-$manifolds. In [11], it was shown to be a quasi-convex subset of $\mathcal{C}(\partial M)$.

By H_g we denote the $3-$dimensional handlebody of genus $g \geq 2$. Recall that a compact connected surface $S \subset H_g$ with boundary is properly embedded if $S \cap \partial H_g = \partial S$ and S is transverse to ∂H_g. A *compressing disk* for S is a properly embedded disk D such that ∂D is essential in S. A properly embedded surface $S \subset H_g$ is *incompressible* if there are no compressing disks for S. Recall also that a

[1] *2010 Mathematics Subject Classification: 57N10, 57N35*

map $F : S \times [0,1] \to H_g$ is a proper isotopy if for all $t \in [0,1]$, $F\left|_{S\times\{t\}}\right.$ is a proper embedding. In this case we will be saying that $F(S \times \{0\})$ and $F(S \times \{1\})$ are properly isotopic in H_g and we will use the symbol $\simeq$ to indicate isotopy in all cases (curves, surfaces etc).

Definition 1 *Let $\mathcal{I}(H_g)$ be the simplicial complex whose vertices are the proper isotopy classes of compressing disks for ∂H_g and of properly imbedded boundary-parallel incompressible annuli and pairs of pants in H_g. For a vertex $[S]$ which is not a class of compressing disks, it is also required that S is isotopic to a surface $\overline{S}$ embedded in ∂H_g via an isotopy*

$$F : S \times [0,1] \to H_g$$

with $F(S \times \{0\}) = S$, $F(S \times \{1\}) = \overline{S}$ and F being proper when restricted to $[0,1)$. A collection of vertices spans a simplex in $\mathcal{I}(H_g)$ when any two of them may be represented by disjoint surfaces in H_g.

Note that the class of properly embedded incompressible surfaces in H_g is very rich. For example, it contains surfaces of arbitrarily high genus (see [13], [3]) which are not included as vertices in the complex $\mathcal{I}(H_g)$ defined above. Also observe that there exist properly embedded annuli and pairs of pants which are not isotopic to a surface entirely contained in ∂H_g. The isotopy classes of such surfaces are also excluded from the vertex set of $\mathcal{I}(H_g)$.

Note that we may regard $\mathcal{D}(H_g)$ as a subcomplex of $\mathcal{I}(H_g)$ or, by taking boundaries of the representative disks, of $\mathcal{C}(\partial H_g)$. Note also that the vertices of $\mathcal{I}(H_g)$ represented by annuli correspond exactly to the vertices of $\mathcal{C}(\partial H_g)$ represented by curves that are essential in ∂H_g but are not meridian boundaries. We define the complex of annuli $\mathcal{A}(H_g)$ to be the subcomplex of $\mathcal{I}(H_g)$ spanned by these vertices. Together, the vertices of $\mathcal{D}(H_g) \cup \mathcal{A}(H_g)$ span a copy of $\mathcal{C}(\partial H_g)$ in $\mathcal{I}(H_g)$, and we regard $\mathcal{C}(\partial H_g)$ as a subcomplex of $\mathcal{I}(H_g)$.

Our goal is to show that for a handlebody H_g of genus $g \geq 2$ the automorphisms of the complex $\mathcal{I}(H_g)$ are all geometric, that is, they are induced by homeomorphisms of H_g. This can be rephrased by saying that the map

$$A : \mathcal{MCG}(H_g) \to Aut(\mathcal{I}(H_g))$$

is an onto map, where $Aut(\mathcal{I}(H_g))$ is the group of automorphisms of the complex $\mathcal{I}(H_g)$ and $\mathcal{MCG}(H_g)$ is the (extended) mapping class group of H_g, i.e. the group of isotopy classes of self-homeomorphisms of H_g. Moreover, we will show (see Theorem 7 below) that the map A is 1-1 except when H_g is the handlebody of genus 2 in which case a $\mathbb{Z}_2$ kernel is present generated by the hyper-elliptic involution.

For the proof of this result we perform a close examination of links of vertices in $\mathcal{I}(H_g)$. This examination establishes that an automorphism f of $\mathcal{I}(H_g)$ must

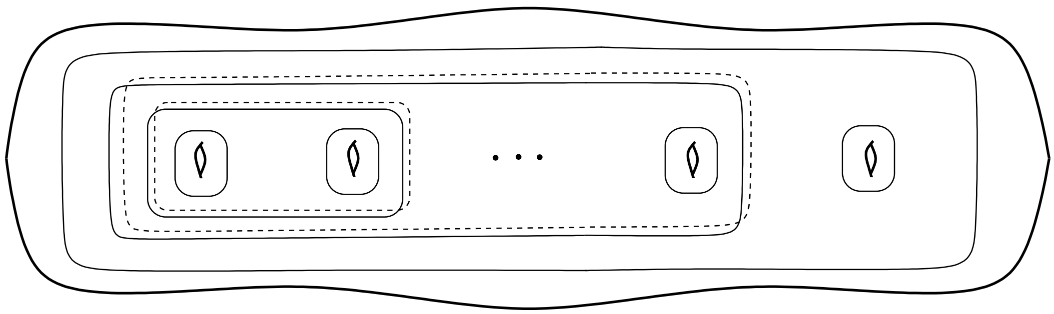

Figure 1: Pants decomposition for H_g consisting of non-separating, non-meridian curves, $g \geq 3$.

map each vertex v in $\mathcal{I}(H_g)$ to a vertex $f(v)$ consisting of surfaces of the same topological type as those in v. In particular, f induces an automorphism of the subcomplex $\mathcal{C}(\partial H_g)$ which permits the use of the corresponding result for surfaces (see [7], [9]).

It is a well known result that for genus ≥ 2 the complex of curves $\mathcal{C}(\partial H_g)$ is a $\delta-$hyperbolic metric space in the sense of Gromov (see [10],[2]). In the last section we deduce that the complex $\mathcal{I}(M)$ is itself a $\delta-$hyperbolic metric space in the sense of Gromov. Moreover, it follows that $Aut(\mathcal{I}(H_g))$ does not contain parabolic elements and the hyperbolic isometries of $\mathcal{I}(M)$ correspond to pseudo-Anosov elements of $\mathcal{MCG}(H_g)$.

In a recent preprint of M. Korkmaz and S. Schleimer (see [8]), it was shown, in a more general context, that $\mathcal{MCG}(H_g)$ and $Aut(\mathcal{D}(H_g))$ are isomorphic. Apart from this isomorphism, our motivation for constructing the copmplex $\mathcal{I}(H_g)$ is the study of the mapping class group of a Heegaard splitting in a 3$-$manifold M. This group (originally defined for $\mathbb{S}^3$ and often called the Goeritz mapping class group) consists of the isotopy classes of orientation preserving homeomorphisms of M that preserve the Heegaard splitting. The mapping class group of a Heegaard splitting is known to be finitely presented (see [1], [4], [14]) only for $M = \mathbb{S}^3$ and for a genus 2 Heegaard splitting. We aim to examine the corresponding open questions for $M = \mathbb{S}^3$ and Heegaard splittings of genus ≥ 3 as well as for certain classes of hyperbolic 3-manifolds. For these purposes, the complex $\mathcal{I}(H_g)$ is a suitable building block for defining a complex encoding the complexity of the Goeritz mapping class group, because $\mathcal{I}(H_g)$ contains a copy of the curve complex of the boundary surface ∂H_g.

1.1 Notation and terminology

A 3$-$dimensional handlebody H_g of genus g can be represented as the union of a handle of index 0 (i.e. a 3$-$ball) with g handles of index 1 (i.e. g copies of

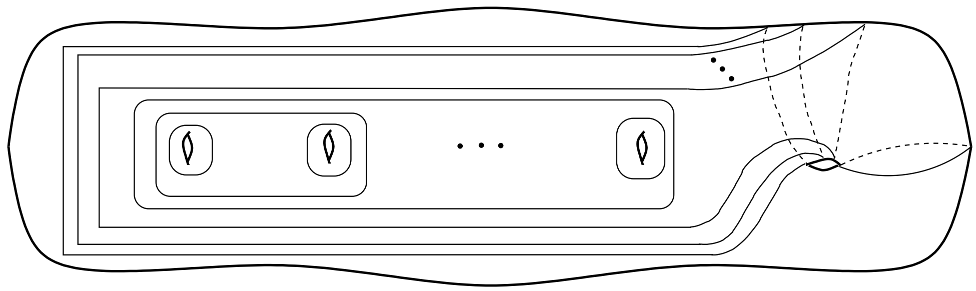

Figure 2: Pants decomposition for H_g consisting of a single non-separating meridian curve, and $3g-4$ non-meridian curves, $g \geq 3$.

$D^2 \times [0,1]$).

For an essential simple closed curve α in ∂H_g we will be writing $[\alpha]$ for its isotopy class and the corresponding vertex in $\mathcal{C}(\partial H_g)$. We will be writing $[S_\alpha]$ for the corresponding vertex in $\mathcal{A}(H_g)$ where S_α is the annulus corresponding to the curve α, provided that α is not a meridian boundary. We will be saying that $[S_\alpha]$ is an *annular vertex.* If α is a meridian boundary we will be writing $[D_\alpha]$ for the corresponding vertex in $\mathcal{D}(H_g)$. We will be saying that $[D_\alpha]$ is a *meridian vertex* and α a meridian curve. A vertex in $\mathcal{I}(H_g) \setminus (\mathcal{D}(H_g) \cup \mathcal{A}(H_g))$ will be called a *pants vertex.*

By writing $[\alpha] \cap [\beta] = \varnothing$ for non-isotopic curves α, β we mean that there exist curves $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$ such that $\alpha' \cap \beta' = \varnothing$. By writing $[\alpha] \cap [\beta] \neq \varnothing$ we mean that for any $\alpha' \in [\alpha]$ and $\beta' \in [\beta]$, $\alpha' \cap \beta' \neq \varnothing$. By saying that the class $[\alpha]$ intersects the class $[\beta]$ at one point we mean that, in addition to $[\alpha] \cap [\beta] \neq \varnothing$, there exist curves $\alpha' \in [a]$ and $\beta' \in [\beta]$ which intersect at exactly one point.

The above notation with square brackets will be similarly used for surfaces. If S is an incompressible surface we will denote by $Lk([S])$ the link of the vertex $[S]$ in $\mathcal{I}(H_g)$, namely, for each simplex σ containing $[S]$ consider the faces of σ not containing $[S]$ and take the union over all such σ. We will use the notation $\ncong$ to declare that two links are not isomorphic as complexes.

We will also use the classical notation $\Sigma_{n,b}$ to denote a surface of genus n with b boundary components.

2 Properties of the complex $\mathcal{I}(H_g)$

In this section we will show that every automorphism of $\mathcal{I}(M)$ must preserve the subcomplexes $\mathcal{A}(H_g)$ and $\mathcal{D}(H_g)$. In particular, we will show that for $[S] \in \mathcal{I}(H_g)$, the topological type of the surface S determines the link of $[S]$ in $\mathcal{I}(H_g)$ and vice-versa. To do this we will find topological properties for the link of each

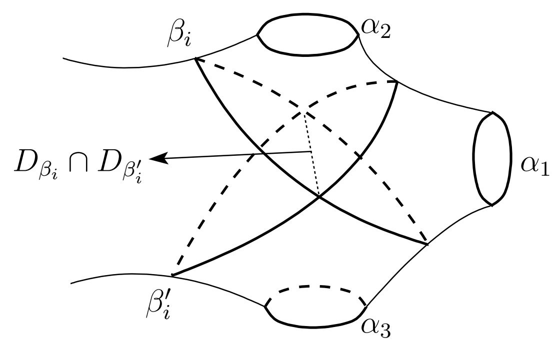

Figure 3:

topological type of surfaces (meridians, annuli and pairs of pants) that distinguish their links.

It is well known that a pants decomposition for ∂H_g is a collection $\alpha_1, \ldots, \alpha_{3g-3}$ of $3g-3$ essential, non-parallel, simple closed curves such that the closure of each component of the complement of these curves is a pair of pants. The number of pairs of pants is $2g-2$. Thus, the maximal number of vertices in a simplex of $\mathcal{I}(H_g)$ is $5g-5$. In other words the dimension of $\mathcal{I}(H_g)$ is $\leq 5g-6$. To see that simplices of dimension $5g-6$ actually exist, observe that there exists a pants decomposition $\alpha_1, \ldots, \alpha_{3g-3}$ so that each α_i is a non-separating, non-meridian curve for all i. This is displayed in Figure 1 for $g \geq 3$ and for $g = 2$ see Remark 6 below. For such a choice of α_i's, all $2g-2$ pairs of pants formed by $\alpha_1, \ldots, \alpha_{3g-3}$ are incompressible surfaces. Apparently, all such pairs of pants give rise to distinct elements in $\mathcal{I}(H_g)$. Thus, a pants decomposition $\alpha_1, \ldots, \alpha_{3g-3}$ with all α_i's being non-meridian curves gives rise to $3g-3$ annular surfaces $S_{\alpha_1}, \ldots, S_{\alpha_{3g-3}}$. These surfaces along with the $2g-2$ pairs of pants formed by $\alpha_1, \ldots, \alpha_{3g-3}$ give rise to a simplex in $\mathcal{I}(H_g)$ containing $5g-5$ vertices. We have established the following

Proposition 2 *The dimension of the complex* $\mathcal{I}(H_g)$ *is* $5g-6$.

We next examine the dimension of $Lk([D])$ when D is a meridian and of $Lk([S_\alpha])$ when S_α is an annular surface.

Lemma 3 *If* S_α *is an annular (incompressible) surface then the link of the vertex* $[S_\alpha]$ *in* $\mathcal{I}(H_g)$ *has dimension* $5g-7$.

Proof. We first assume that α is a separating curve. Then α decomposes ∂H_g into surfaces $\Sigma_{n,1}$ and $\Sigma_{m,1}$ with $m+n=g$ and $m, n \geq 1$ with α being isotopic to the boundary of $\Sigma_{n,1}$ as well as to the boundary of $\Sigma_{m,1}$. To complete the proof in this case, it suffices to find a pants decomposition for ∂H_g consisting

of non-meridian curves and containing the curve α. For the latter, it suffices to show the following

Claim $\Sigma_{n,1}$ can be decomposed into $2n-1$ pairs of pants so that the boundary curves of each are non-meridian when viewed as curves in ∂H_g.

The first step is to find pair-wise disjoint non-separating curves $\alpha_1, , \ldots, \alpha_n$ in $\Sigma_{n,1}$ such that α_i does not bound a disk in H_g for all i. To see this, let α_1, α_1' be two simple non-separating curves in ∂H_g such that the curves $\alpha, \alpha_1, \alpha_1'$ bound a pair of pants in ∂H_g. As α is not the boundary of a meridian in H_g, it is clear that α_1, α_1' cannot both be meridian boundaries in H_g. Assuming α_1 is not meridian boundary, we may cut $\Sigma_{n,1}$ along α_1 to obtain a surface $\Sigma_{n-1,3}$. By the same argument, we may find a non-separating curve α_i in $\Sigma_{n-(i-1),2i-1}$, $i = 2, \ldots, n$ which is not meridian boundary.

Apparently, cutting $\Sigma_{n,1}$ along $\alpha_1, , \ldots, \alpha_n$ we obtain a sphere $\Sigma_{0,1+2n}$ with $1+2n$ holes, such that the boundary components of $\Sigma_{0,1+2n}$ do not bound disks when viewed as curves in ∂H_g. We now claim that we may find pair-wise disjoint curves $\beta_1, \ldots, \beta_{2n-2}$ such that β_j does not bound a disk in H_g for all $j = 1, \ldots, 2n-2$. To see this, let β_1, β_1' be two simple closed curves in $\Sigma_{0,1+2n}$ such that the curves $\alpha_1, \alpha_2, \beta_1$ bound a pair of pants and the curves $\alpha_1, \alpha_3, \beta_1'$ bound a pair of pants as shown in Figure 3. If both β_1, β_1' bound properly embedded disks in H_g, say $D_{\beta_1}, D_{\beta_1'}$ respectively, then $D_{\beta_1} \cap D_{\beta_1'}$ is a properly embedded arc in H_g which separates D_{β_1} into two half-disks. Similarly for $D_{\beta_1'}$. Appropriate unions of these half-disks along $D_{\beta_1} \cap D_{\beta_1'}$ establish a contradiction since none of $\alpha_1, \alpha_2, \alpha_3$ is a meridian boundary. Thus, at least one of β_1, β_1', say β_1, does not bound a disk. Cutting $\Sigma_{0,1+2n}$ along β_1 we obtain a pair of pants and a surface $\Sigma_{0,1+2n-1}$ which has the same property as $\Sigma_{0,1+2n}$, namely, all boundary components of $\Sigma_{0,1+2n-1}$ do not bound disks when viewed as curves in ∂H_g. By applying the same argument repeatedly, we may find the desired collection of curves $\beta_1, \ldots, \beta_{2n-2}$ none of which is a meridian boundary. Apparently, the collection of curves $\beta_1, \ldots, \beta_{2n-2}$ decomposes $\Sigma_{0,1+2n}$ into $2n-1$ pairs of pants as required. This completes the proof of the Claim and the proof of the lemma in the case α is separating.

Assume now that α is non-separating. Using two copies of α and a simple arc joining them we may construct a separating curve β which decomposes ∂H_g into surfaces $\Sigma_{g-1,1}$ and $\Sigma_{1,1}$ with β being isotopic to the boundary of $\Sigma_{g-1,1}$ as well as to the boundary of $\Sigma_{1,1}$. Note that $\Sigma_{1,1}$ contains α. Then by the above claim we have that $\Sigma_{g-1,1}$ can be decomposed into $2(g-1)-1$ (incompressible) pairs of pants by using non-meridian curves α_i, $i = 1, \ldots, 3g-5$ contained in $\Sigma_{g-1,1}$ together with the cirve β. By adding the curve α we obtain a pants decomposition $\alpha_1, \ldots, \alpha_{3g-5}, \beta, \alpha$ with all curves being non-meridian. Hence, $[S_\alpha]$ is contained in a simplex of maximum dimension, namely, of dimension $5g-6$ which shows that the dimension of $Lk([S_\alpha])$ is $5g-7$. $\blacksquare$

Lemma 4 *If D is a meridian then the link of the vertex $[D]$ in $\mathcal{I}(H_g)$ has dimension $5g-9$.*

Proof. First assume that $[D]$ is non-separating. We may find a pants decomposition $\alpha_1, \ldots, \alpha_{3g-4}, \alpha_{3g-3} = \partial D$ for ∂H_g such that α_i is non-meridian for all $i = 1, \ldots, 3g-4$ (see Figure 2). This collection of curves decomposes ∂H_g into $2g-2$ pairs of pants such that exactly two of these have ∂D as boundary component and, hence, they are compressible surfaces. Thus, a non-separating meridian $[D]$ is contained in a simplex with $3g-3+2g-4$ vertices and, hence, the dimension of $Lk([D])$ is $\geq 5g-9$. Let now $\alpha_1', \ldots, \alpha_{3g-4}', \alpha_{3g-3} = \partial D$ be any pants decomposition with corresponding pairs of pants $P_1, \ldots, P_{2g-2}$ such that one of them, say P_1, has two boundary components isotopic to ∂D. Then the third boundary component of P_1 will also be a meridian, thus, another pair of pants distinct from P_1 will also be compressible. This shows that a class $[D]$ with D non-separating meridian cannot be contained in a simplex of more that $5g-7$ vertices and, thus, $Lk([D])$ is equal to $5g-9$.

If $[D]$ is separating, it is clear that any decomposition $\alpha_1, \ldots, \alpha_{3g-4}, \alpha_{3g-3} = \partial D$ for ∂H_g with α_i being non-meridian for all $i = 1, \ldots, 3g-4$ has the property that exactly two of the corresponding pairs of pants are compressible and we work similarly. ■

Proposition 5 *Let $[D]$ be a meridian vertex, $[S_\alpha]$ an annular vertex and $[P]$ a pants vertex. Then the links $Lk([D]), Lk([S_\alpha])$ and $Lk([P])$ are pair-wise non-isomorphic as complexes.*

Proof. By the previous two Lemmata, the links of the vertices $[D]$ and $[S_\alpha]$ have distinct dimensions, hence, it is clear that $Lk([D]) \ncong Lk([S_\alpha])$. It remains to distinguish $Lk([P])$ from $Lk([D])$ and $Lk([S_\alpha])$.

Let $[P]$ be a vertex in $\mathcal{I}(M)$ such that P is a pair of pants with boundary components β, γ, δ. The vertices in $Lk([P])$ form a cone graph, that is, the vertex $[S_\beta]$ belongs to $Lk([P])$ and is connected by an edge with any vertex in $Lk([P])$. We will reach a contradiction by showing that

$$\forall [Q] \in Lk([D]), \exists [R] \in Lk([D]) : [Q] \cap [R] \neq \varnothing \qquad (*)$$

and similarly for $Lk([S_\alpha])$. For, if β_Q is a boundary component of a surface representing $[Q] \in Lk([D])$ then there exists a curve γ such that $\partial D \cap \gamma = \varnothing$ and $\gamma \cap \beta_Q \neq \varnothing$. Let $[R]$ be the vertex represented by S_γ if γ is non-meridian and by D_γ if γ is a meridian boundary. Then $[R] \in Lk([D])$ is the required vertex which is not connected by an edge with $[Q]$, thus $Lk([D])$ satisfies property $(*)$. Similarly, we show that $Lk([S_\alpha])$ also satisfies property $(*)$. ■

Remark 6 *Let α, β, γ be non-separating curves in ∂H_2 decomposing ∂H_2 into two components which we denote by P, P'. Note that P, P' may not be isotopic. To*

see this, denote by f_1, f_2 the generators of $\pi_1(H_2)$ corresponding to the longitudes of H_2. We may choose non-separating curves α, β on ∂H_2 which represnt the second powers f_1^2, f_2^2 up to conjugacy. Choose an essential non-separating curve γ such that α, β, γ are mutually disjoint and non isotopic. These curves separate ∂H_2 into two components (pairs of pants) P and P'. If P, P' were isotopic then H_2 would be homeomorphic to the product $P \times [0,1]$ and any two of the boundary components of P would give rise to generators for $\pi_1(H_2)$. Since neither $\alpha \simeq f_1^2$ nor $\beta \simeq f_2^2$ are generators for the free group on f_1, f_2 it follows that, for this particular choice of α, β, γ, the surfaces P, P' are not isotopic.

3 Proof of the Main Theorem

Let

$$A : \mathcal{MCG}(H_g) \to Aut(\mathcal{I}(H_g))$$

be the map sending a mapping class F to the automorphism it induces on $\mathcal{I}(H_g)$, that is, $A(F)$ is given by

$$A(F)[S] := [F(S)].$$

Theorem 7 *The map $A : \mathcal{MCG}(H_g) \to Aut(\mathcal{I}(H_g))$ is onto for $g \geq 2$ and injective for $g \geq 3$. For $g = 2$, A has a $\mathbb{Z}_2-$kernel generated by the hyper-elliptic involution.*

We will use the following immediate Corollary of Proposition 5.

Corollary 8 *Automorphisms of $\mathcal{I}(H_g)$ preserve all types (meridian, annular and pants) of vertices.*

We will also need the following

Lemma 9 *If $f \in Aut(\mathcal{I}(H_g))$ and $f|_{\mathcal{D}(H_g)\cup\mathcal{A}(H_g)} = id_{\mathcal{D}(H_g)\cup\mathcal{A}(H_g)}$ then $f([S]) = [S]$ for any vertex $[S] \in \mathcal{I}(M)$ except in the case mentioned in Remark 6, namely, if $g = 2$ and P is a pair of pants with all boundary components of ∂P being separating curves decomposing ∂H_2 into 2 components P, P', then either, $f([P]) = [P]$ or, $f([P]) = [P']$.*

Proof. We have to show that $f \in Aut(\mathcal{I}(H_g))$ fixes every vertex $[P]$ where P is a pair of pants. Let $[P]$ be such a vertex in $\mathcal{I}(H_g)$. By Corollary 8 it is clear that $f([P])$ is a vertex $[P']$ with P' being a pair of pants. Denote by α_1, α_2, α_3 the boundary components of P and, similarly, α_1', α_2', α_3' for P'. If $[\alpha_{i_0}] \cap [\alpha_{j_0}'] \neq \varnothing$ for some $i_0, j_0 \in \{1,2,3\}$ then the vertex $[S_{\alpha_{i_0}}]$ is connected by an edge with $[P]$

and is not connected by an edge with $[P']$. As $\left[S_{\alpha_{i_0}}\right]$ is fixed by f, it follows that $f([P])$ cannot be equal to $[P']$. Thus, we may assume that

$$[\alpha_i] \cap \left[\alpha'_j\right] = \varnothing \text{ for all } i, j = 1, 2, 3. \qquad (**)$$

Consider the following property:

$$\text{Up to change of enumeration, } \alpha_i \simeq \alpha'_i \text{ for } i = 1, 2, 3. \qquad (***)$$

If property $(***)$ holds then $P \simeq P'$ unless $g = 2$ and α_1, α_2, α_3 are all non-separating curves which decompose ∂H_2 into 2 pairs of pants (cf. Remark 6) which may or may not be isotopic. Thus, if property $(***)$ holds then either $f([P]) = [P]$ or the exception in the statement of the lemma occurs.

We examine now the case where $g \geq 3$ and property $(***)$ does not hold. By assumption $(**)$, we may cut ∂H_g along α_1, α_2, α_3 to obtain either

- the surface P and a surface $\Sigma_{g-2,3}$ (if all α_1, α_2, α_3 are non-separating) or,
- the surface P, a surface $\Sigma_{g_1,1}$ and a surface $\Sigma_{g-g_1-1,2}$ for some $0 < g_1 < g$ (if exactly one of α_1, α_2, α_3 is separating and the other two curves are non-isotopic) or,
- the surface P and a surface $\Sigma_{g-1,1}$ (if exactly one of α_1, α_2, α_3 is separating and the other two curves are isotopic) or,
- the surface P and surfaces $\Sigma_{g_1,1}, \Sigma_{g_2,1}, \Sigma_{g_3,1}$ for some $g_1, g_2, g_3 \geq 1$ with $g_1 + g_2 + g_3 = g$ (if all α_1, α_2, α_3 are separating)

Note that if P is a pair of pants, it is impossible to have exactly two of its boundary curves α_1, α_2, α_3 being separating. In all cases, P' is contained in a surface of the form $\Sigma_{g',b}$ for some $g' \in \{1, \dots, g-1\}$ and $b \in \{1, 2, 3\}$ mentioned above. Thus, we may find a non-meridian curve α in ∂H_g such that

$$\alpha \cap \alpha_i = \varnothing, \forall i = 1, 2, 3 \text{ and } [\alpha] \cap \left[\alpha'_{j_0}\right] \neq \varnothing \text{ for some } j_0 \in \{1, 2, 3\}.$$

Then, for the annular surface S_α we have that $[S_\alpha]$ is connected by an edge with $[P]$ and is not connected by an edge with $[P']$. As $[S_\alpha]$ is fixed by f, it follows that $f([P])$ cannot be equal to $[P']$. This completes the proof of the lemma. ■

Proof of Theorem 7. We will use the corresponding result for surfaces, see [7],[9], which applies to the boundary of the handlebody ∂H_g.

We first show that every $f \in Aut(\mathcal{I}(H_g))$ is geometric. By Proposition 5 we know that $f(\mathcal{A}(H_g)) = \mathcal{A}(H_g)$ and $f(\mathcal{D}(H_g)) = \mathcal{D}(H_g)$. In particular, $f(\mathcal{C}(\partial H_g)) = \mathcal{C}(\partial H_g)$. The restriction $f|_{\mathcal{C}(\partial H_g)}$ of f on $\mathcal{C}(\partial H_g)$ induces an automorphism of $\mathcal{C}(\partial H_g)$ which by the analogous result for surfaces (see [7],[9]) is geometric, that is, there exists a homeomorphism

$$F_{\partial H_g} : \partial H_g \to \partial H_g$$

such that $A\left(F_{\partial H_g}\right) = f|_{\mathcal{C}(\partial H_g)}$. As $f|_{\mathcal{C}(\partial H_g)}$ maps $\mathcal{D}(M)$ to $\mathcal{D}(M)$, $F_{\partial H_g}$ sends meridian boundaries to meridian boundaries. It follows that $F_{\partial H_g}$ extends to a homeomorphism $F : H_g \to H_g$. We know that $A(F) = f$ on $\mathcal{C}(\partial H_g)$ and we must show that $A(F) = f$ on $\mathcal{I}(H_g)$. This follows from Lemma 9 which completes the proof that every $f \in Aut(\mathcal{I}(H_g))$ is geometric.
Let $f \in Aut(\mathcal{I}(H_g))$. Since A is shown to be onto, there exists a homeomorphism $F : H_g \to H_g$ such that $A([F]) = f$. This implies that $f(\mathcal{D}(H_g)) = \mathcal{D}(H_g)$ and $f(\mathcal{A}(H_g)) = \mathcal{A}(H_g)$. In particular, f restricted to $\mathcal{C}(\partial H_g) \equiv \mathcal{D}(H_g) \cup \mathcal{A}(H_g)$ induces an automorphism $\overline{f}$ of the complex of curves $\mathcal{C}(\partial H_g)$. By [7], [9], there exists a homeomorphism $F_{\partial H_g} : \partial H_g \to \partial H_g$ such that $A\left(F_{\partial H_g}\right) = \overline{f}$. Such a homeomorphism is unique unless $g = 2$ in which case the map

$$\mathcal{MCG}(\partial H_2) \to Aut(\mathcal{C}(\partial H_2))$$

has a $\mathbb{Z}_2-$kernel generated by an involution of ∂H_2. However, any homeomorphism of ∂H_g which extends to H_g it does so uniquely (see, for example, [5, Theorem 3.7 p.94]), and therefore the map

$$\mathcal{MCG}(H_g) \to Aut(\mathcal{I}(H_g))$$

is injective unless $g = 2$ in which case it has a $\mathbb{Z}_2-$kernel. ∎

4 Applications

We first establish hyperbolicity for $\mathcal{I}(H_g)$.

Proposition 10 *The complex $\mathcal{I}(H_g)$ is $\delta-$hyperbolic in the sense of Gromov.*

Proof. As far as hyperbolicity is concerned, the 1-skeleton $\mathcal{I}(H_g)^{(1)}$ of $\mathcal{I}(H_g)$ is relevant. $\mathcal{I}(H_g)^{(1)}$ is endowed with the combinatorial metric so that each edge has length 1. Apparently, we have an embedding

$$i : \mathcal{C}(\partial H_g)^{(1)} \hookrightarrow \mathcal{I}(H_g)^{(1)}$$

with $i\left(\mathcal{C}(\partial H_g)^{(1)}\right) = \mathcal{D}(H_g)^{(1)} \cup \mathcal{A}(H_g)^{(1)}$ where the superscript $^{(1)}$ always denotes $1-$skeleton. We claim that this embedding is isometric. Indeed, if $[\alpha_1], [\alpha_2]$ are distinct vertices with distance $d_{\mathcal{C}}([\alpha_1], [\alpha_2])$ in $\mathcal{C}(\partial H_g)^{(1)}$ then the distance $d_{\mathcal{I}}(i([\alpha_1]), i([\alpha_2]))$ cannot be smaller. For, if $[S_0] = i([\alpha_1]), [S_1], \ldots, [S_k] = i([\alpha_2])$ is a sequence of vertices which gives rise to a geodesic in $\mathcal{I}(M)^{(1)}$ of length less than $d_{\mathcal{C}}([\alpha_1], [\alpha_2])$, equivalently,

$$d_{\mathcal{I}}(i([\alpha_1]), i([\alpha_2])) = k < d_{\mathcal{C}}([\alpha_1], [\alpha_2])$$

then for each $j = 1, 2, \ldots, k-1$ consider β_j to be any boundary component of S_j. It is clear that β_j is disjoint from β_{j-1} and β_{j+1}. Therefore, the sequence $[\alpha_1], [\beta_1], \ldots, [\beta_{k-1}], [\alpha_2]$ is a segment in $\mathcal{C}(\partial H_g)^{(1)}$ of length k with $k < d_{\mathcal{C}}([\alpha_1], [\alpha_2])$, a contradiction.

For any vertex $[P]$ in $\mathcal{I}(H_g)^{(1)} \setminus \mathcal{D}(H_g)^{(1)} \cup \mathcal{A}(H_g)^{(1)}$ we may find an annular vertex, namely, $[S_{\partial P}]$ where ∂P is any component of the boundary of P, which is connected by an edge with $[P]$. Thus, $\mathcal{I}(H_g)^{(1)}$ is within bounded distance from $i\left(\mathcal{C}(\partial H_g)^{(1)}\right)$. Since $\mathcal{C}(\partial H_g)^{(1)}$ is $\delta-$hyperbolic in the sense of Gromov, so is $\mathcal{I}(H_g)^{(1)}$. ■

An element $F \in \mathcal{MCG}(H_g)$ is called pseudo-Anosov when it restricts to a pseudo-Anosov homeomorphism on ∂H_g. The proof of the following proposition is immediate from the corresponding result for surfaces (see [10, Prop. 4.6]) along with the above mentioned fact that $\mathcal{C}(\partial H_g)$ is cobounded in $\mathcal{I}(H_g)$.

Proposition 11 *For any $g \geq 2$, there exists a $c > 0$ such that any pseudo-Anosov $F \in \mathcal{MCG}(H_g)$, any vertex $v \in \mathcal{I}(H_g)$ and any $n \in \mathbb{Z}$,*

$$d_{\mathcal{I}}(F^n(v), v) \geq c|n|.$$

Thus, pseudo-Anosov elements in $\mathcal{MCG}(H_g)$ correspond to hyperbolic isometries of $\mathcal{I}(H_g)$ and there are no parabolic isometries for $\mathcal{I}(H_g)$.

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